# Note: Pairwise Rearrangements in Reliability Structures 

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#### Abstract

Boland, Proschan, and Tong [2] used the notion of criticality of nodes in a coherent system to study the optimal component arrangement of reliability structures. They also provided a sufficient minimal cut (path) based criterion for verifying the criticality ordering of two nodes. We develop a necessary and sufficient condition for two nodes to be comparable and provide specific examples illustrating our result's applicability. As a corollary, certain optimal arrangement properties of well-known systems are derived. © 1994 John Wiley \& Sons, Inc.


## 1. INTRODUCTION

The optimal component arrangement in a coherent structure is of interest in the design and assembly of reliability networks. Roughly speaking, the optimal arrangement problem is the allocation of components to the various positions in the system so as to increase or maximize reliability or some other system performance characteristic. For a series, parallel, $k$-out-of- $n$ system and parallel-series, series-parallel systems, the optimal assignment was studied in [4] and [7], respectively. Recently, Derman, Lieberman, and Ross [5] stated their famous conjecture on the optimal arrangement of a " 2 -consecutive failures-out-of- $n$ fails system," which was proved by Malon [12] (see also [6, 13, 20]).

In a recent article by Boland et al. [2], a new method was developed for obtaining the optimal permutation by a process of elimination. The process depends on a notion of criticality of the nodes of a coherent system. Their main result states that, if node $i$ is more critical than node $j$ and a less reliable component is assigned to node $i$, then an improvement is made by interchanging the components assigned to nodes $i$ and $j$. Some interesting results of similar flavor can be found also in [14].

For the comparison of the criticality of two nodes, Boland et al. [2] proposed a criterion based on the cut (path) sets of the structure. Their result does not provide a necessary and sufficient condition and, as they point out, there are cases where orderable nodes cannot be detected by it.

In this article we provide a necessary and sufficient condition for the comparison of two nodes with respect to their criticality. Our main result is used for establishing an easy proof of the transitivity of the criticality ordering property. Finally, some illustrative examples are presented; in most of them, the criterion of Boland et al. [2] fails, whereas ours offers a tool for a further study of the optimal component assignment.

## 2. OPTIMAL ARRANGEMENT AND CRITICALITY OF NODES

Consider a coherent system of $n$ components. Using the notation of Barlow and Proschan [1], we let $\phi$ denote the structure function of the system, $\mathbf{x}=\left(x_{1}, x_{2}, \ldots\right.$, $\left.\boldsymbol{x}_{n}\right)$ a state vector, and $\mathbf{p}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ the vector of component reliabilities. We
assume that each one of the $n$ components can be placed in any one of the $n$ positions (nodes) of the system. A choice of such a placement results in a permutation $\pi$ of the $n$ numbers [i.e., the $i$ th node of the system will be occupied by the component $\pi(i)$ ( $i=1,2, \ldots, n)$. Denoting by $R(\mathbf{p} ; \pi)$ the reliability of the system under the permutation $\pi$, we define the following.

DEFINITION 2.1: The permutation $\boldsymbol{\pi}_{0}$ is said to be optimal if and only if

$$
R\left(\mathbf{p} ; \boldsymbol{\pi}_{0}\right)=\max _{\pi} R(\mathbf{p} ; \boldsymbol{\pi}),
$$

where the maximum is taken over the $n!$ permutations of $\{1,2, \ldots, n\}$.
In the sequel, without loss of generality, we assume that $0<p_{1} \leq p_{2} \leq \cdots \leq p_{n}<1$.
A particularly interesting question is whether the permutation maximizing the reliability is independent of the particular choices of $p_{1}, p_{2}, \ldots, p_{n}$. In this article we assume knowledge of the ranks of the component reliabilities and not of their actual values. Such a situation occurs, for example, when the components have decreasing (or increasing) failure rates and their ages are known. In this case the component reliabilities can be ordered according to their ages. Following the notation used in [2], let ( $1_{i}, 0_{j}$, $\mathbf{x}^{(i, j)}$ ) be a state vector for which $x_{i}=1, x_{j}=0$, and $\mathbf{x}^{(i, j)}$ is the vector obtained by deleting the $i$ th and $j$ th coordinate in $\mathbf{x}$. The vector $\left(0_{i}, 1_{j}, \mathbf{x}^{(i, j)}\right)$ is defined similarly. The following definition is due to Boland et al. [2].

DEFINITION 2.2: Node $i$ is more critical than node $j$ for the structure function $\phi$ (notation: $i>j$ ) if and only if

$$
\begin{equation*}
\phi\left(1_{i}, 0_{j}, \mathbf{x}^{(i, j)}\right) \geq \phi\left(0_{i}, 1_{j}, \mathbf{x}^{(i, j)}\right) \tag{1}
\end{equation*}
$$

holds for all $\mathbf{x}^{(i, j)}$ and strict inequality is true for some $\mathbf{x}^{(i, j)}$.
For any permutation $\pi$ let us denote by $\pi_{i j}$ the pairwise rearrangement permutation obtained by interchanging the $i$ th and $j$ th coordinates. Boland et al. [2] proved that if $\pi(i)<\pi(j)$, then the next theorem holds true.

THEOREM 2.1: Node $i$ is more critical than node $j$ if and only if $R\left(\mathbf{p} ; \boldsymbol{\pi}_{i j}\right) \geq R(\mathbf{p}$; $\boldsymbol{\pi})$ for all $\mathbf{p}$, with strict inequality for some $\mathbf{p}$.

Theorem 2.1 gives rise to a procedure for obtaining the optimal permutation of a given coherent structure by a process of elimination of inadmissible permutations via pairwise permutations; for details see [2].

Because the examination of the criticality ordering of two nodes by use of Definition 2.2 is, in general, laborious, especially for large or structurally complicated systems, Boland et al. [2] proposed the next criterion based on the minimal cut or path sets of the network.

THEOREM 2.2: Let $A_{i}$ be the set of all minimal cut (path) sets of the structure $\phi$ containing node $i$. If $A_{j} \subset A_{i}$ then $i \stackrel{\substack{s}}{ }$.

The converse of Theorem 2.2 is not true. In consequence, there are cases where the criticality ordering of specific nodes cannot be detected and the mission of establishing optimal arrangements remains incomplete. The next theorem, which is also based on the cut (path) sets of the structure, provides a necessary and sufficient condition.

THEOREM 2.3: If $i, j \in I=\{1,2, \ldots, n\}$ the following are equivalent:
(a) $i s j$.
(b) For every $S \subseteq I-\{i, j\}$ such that $S \cup\{j\}$ is a cut set, the set $S \cup\{i\}$ is a cut set too. Moreover, there exists $S_{0} \subseteq 1-\{i, j\}$ such that $S_{0} \cup\{i\}$ is a cut set while $S_{0} \cup\{j\}$ is not.
(c) For every $S \subseteq \bar{I}-\{i, j\}$ such that $S \cup\{j\}$ is a path set, the set $S \cup\{i\}$ is a path set too. Moreover, there exists $S_{0} \subseteq I-\{i, j\}$ such that $S_{0} \cup\{i\}$ is a path set while $S_{0} \cup\{j\}$ is not.
PROOF: Assume first that $i>j$ and consider an index set $S \subseteq I-\{i, j\}$ such that $S \cup\{j\}$ is a cut set. Introducing a state vector $\mathbf{x}$ such that $x_{k}=0$ for all $k \in S$ we obtain $\phi\left(1_{i}, 0_{j}, \mathbf{x}^{(i, j)}\right)=0$ and by virtue of (1) we deduce that $\phi\left(0_{i}, 1_{j}, \mathbf{x}_{\varepsilon}^{(i . j)}\right)=0$. This proves that $S \cup\{i\}$ is a cut set of the structure. In addition, because $i>^{c} j$ there exists (by Definition 2.2) a state vector $\boldsymbol{\xi}$ such that $1=\phi\left(1_{i}, 0_{j}, \boldsymbol{\xi}^{(i, j)}\right)>\phi\left(0_{i}, 1_{j}, \boldsymbol{\xi}^{(i, j)}\right)=0$. Obviously $S_{0} \cup\{i\}=\left\{k: \xi_{k}^{(i j)}=0\right\} \cup\{i\}$ is a cut set and $S_{0} \cup\{j\}$ is not. Thus (a) is sufficient for (b).

Assume next that (b) is true and let $\mathbf{x}$ be any state vector. If $\phi\left(1_{i}, 0_{j}, \mathbf{x}^{(i, j)}\right)=1$, then (1) is valid. If $\phi\left(1_{i}, 0_{j}, \mathbf{x}^{(i, j)}\right)=0$, then $S \cup\{j\}=\left\{k: x_{k}^{(i, j)}=0\right\} \cup\{j\}$ is a cut set. Therefore $S \cup\{i\}$ is also a cut set, which implies that $\phi\left(0_{i}, 1_{j}, \mathbf{x}^{(i, j)}\right)=0$. Hence (1) is again true. Moreover, if $S_{0}$ is a subset of $I-\{i, j\}$ such that $S_{0} \cup\{i\}$ is a cut set and $S_{0} \cup\{j\}$ is not, considering a state vector $\boldsymbol{\xi}$ with $\xi_{k}=0$ for all $k \in S_{0}$ we deduce $1=\phi\left(1_{i}, 0_{j}, \boldsymbol{\xi}^{(i, j)}\right)>$ $\phi\left(0_{i}, 1_{j}, \xi^{(i, j)}\right)=0$. Thus (b) is sufficient for (a).

In a similar fashion, one can easily verify that condition (c) is necessary and sufficient for (a).

COROLLARY 2.1: If $i>^{\mathcal{c}} j$ and $j \stackrel{c}{>} k$ then $i{ }^{\stackrel{c}{>} k} k$.
PROOF: Let $S \subseteq I-\{i, k\}$ such that $S \cup\{k\}$ is a cut set. If $j \notin S$, then applying twice Theorem 2.3 we deduce that $S \cup\{j\}$ is a cut set (since $j{ }^{c}+k$ ) and $S \cup\{i\}$ is also a cut set (since $i>j$ ). If $j \in S$, we may write $S \cup\{k\}=\left(S_{1} \cup\{k\}\right) \cup\{j\}$ with $S_{1} \cup\{k\}$ $\subseteq I-\{i, j\}$ and the assumption $i{ }^{c} j$ implies that $\left(S_{1} \cup\{k\}\right) \cup\{i\}$ is a cut set. Moreover, because $\left(S_{1} \cup\{k\}\right) \cup\{i\}=\left(S_{1} \cup\{i\}\right) \cup\{k\}$ with $S_{1} \cup\{i\} \subseteq I-\{j, k\}$ and $j{ }^{c}=k$, we conclude that $\left(S_{1} \cup\{i\}\right) \cup\{j\}=S \cup\{i\}$ is a cut set. The rest of condition (b) of Theorem 2.3 can be easily checked using a similar reasoning.

It should be mentioned that the proof of Corollary 2.1 could be argued directly from Definition 2.2 by carrying out a pivotal decomposition (see [1]).

For certain structure functions $\phi$, the family of minimal cut (path) sets, which is substantially smaller than the family of cut (path) sets of the structure, is easy to find. In these cases one can use the next theorem (instead of Theorem 2.3) to compare two nodes with respect to their criticality.

THEOREM 2.4: If $i, j \in I=\{1,2, \ldots, n\}$ the following are equivalent:
(a) $i s j$.
(b) For every $C \subseteq I-\{i, j\}$ such that $C \cup\{j\}$ is a minimal cut set, the set $C \cup\{i\}$ is a cut set. Moreover, there exists an $S_{0} \subseteq i-\{i, j\}$ such that $S_{0} \cup\{i\}$ is a cut set and $S_{0} \cup\{j\}$ is not.
(c) For every $C \subseteq I-\{i, j\}$ such that $C \cup\{j\}$ is a minimal path set, the set $C \cup\{i\}$ is a path set. Moreover, there exists an $S_{0} \subseteq I-\{i, j\}$ such that $S_{0} \cup\{i\}$ is a path set and $S_{0} \cup\{j\}$ is not.

PROOF: In view of Theorem 2.3 it suffices to verify that condition (b) [or (c)] of Theorem 2.4 implies condition (b) [or (c)] of Theorem 2.3. Let $S \subseteq I-\{i, j\}$ such that $S \cup\{j\}$ is a cut set but not minimal. Then there exists a $C_{1} \subset S \cup\{j\}$ such that $C_{1}$ is minimal cut set. If $C_{1}$ does not contain $j$, then $C_{1} \subseteq C_{1} \cup\{i\} \subseteq S \cup\{i\}$, and since $C_{1}$ is a cut set, we conclude that $S \cup\{i\}$ is cut set too. Assume next that $C_{1}$ contains $j$; that is, $C_{1}=C \cup\{j\}$ with $C \subseteq S$. Because $C_{1}=C \cup\{j\}$ is minimal cut set and $C \subseteq I-$ $\{i, j\}$, assumption (b) of Theorem 2.1 ensures that $C \cup\{i\}$ is a cut set; thus $S \cup\{i\} \supseteq C$ $\cup\{i\}$ will also be a cut set. This completes the proof.

Observe that although $C \cup\{j\}$ is a minimal cut (path), the set $C \cup\{i\}$ need not be minimal (e.g., for $\phi\left(x_{1}, x_{2}, x_{3}\right)=x_{1}\left(1-\left(1-x_{2}\right)\left(1-x_{3}\right)\right.$ ) (see [2, p. 808]) observe that $1 \stackrel{c}{>} 2$ and apply Theorem 2.4 for $C=\{3\}$ ).

## 3. APPLICATIONS

We present here some illustrative examples where Theorem 2.4 can be applied.
A. Consider the system described by the structure function (see [2, p. 811])

$$
\phi\left(x_{1}, x_{2}, \ldots, x_{6}\right)=x_{1}\left[1-\left(1-x_{2}\right)\left(1-x_{3}\right)\right]\left[1-\left(1-x_{4}\right)\left(1-x_{5}\right)\left(1-x_{6}\right)\right] .
$$

Theorem 2.2 fails to prove the relation $1 \stackrel{\mathcal{c}}{>} 2$, because the minimam cut set $\{2,3\}$ does not contain component 1 . On the contrary, applying Theorem 2.4 , the validity of $1 \stackrel{c}{>}$ 2 can be easily verified by observing that the only candidate for $C$ is $\{3\}$, leading to $C \cup$ $\{1\}=\{3,1\}$, which is also a cut set.
B. For the consecutive- $k$-out-of- $n$ system (see [3, 10, 12] ) we have $(k<n / 2) k \stackrel{c}{>}$ $k-1 \stackrel{c}{>} \cdots \stackrel{c}{>} 2 \stackrel{c}{>} 1$ and $n-k+1 \stackrel{y}{>} n-k+2>\cdots>n$ (see also [2]). A direct application of Theorem 2.4 shows that the intermediate components $k \leq i<j \leq n-$ $k+1$ are incomparable (obviously this result could not be derived through Theorem 2.2).

In a circular consecutive- $k$-out-of- $n$ system (see $[3,5,15]$ ), Theorem 2.4 shows that, no pair of nodes can be ordered via criticality.
C. For the $r$-within-consecutive- $k$-out-of- $n$ system (see $[9,15,18,19]$ ) Papastavridis and Sfakianakis [16] proved $i+1{ }^{c} i$ for $1 \leq i \leq \min (k-1, n-k)$ and $i \stackrel{c}{>} i+1$ for $\max (k, n-k+1) \leq i \leq n-1$ and pointed out that these relations cannot be established through Theorem 2.2. Nevertheless they could be easily verified, by making use of our Theorem 2.4. Moreover, the intermediate components $\min (k, n-k+$ $1) \leq i<j \leq \max (k, n-k+1)$ are incomparable.
D. The two-dimensional consecutive- $k$-out-of- $n$ : $F$ system, which was introduced by Salvia and Lasher [17] (see also [8, 11]), consists of $n^{2}$ components arranged on a square grid of size $n$ and fails, if and only if there exists at least one square grid of size $k$ ( $2 \leq$ $k \leq n-1$ ) that contains all failed components. Any square grid of size $k$ is a minimal cut set, and applying Theorem 2.4, we immediately obtain the following:

1. If $1 \leq l<\min (k, n-k+1), 1 \leq i, j \leq n$, then $(l+1, j)>(l, j)$ and $(i, l+1)>(i, l)$.
2. If $\max (k, n-k+1) \leq l<n, 1 \leq i, j \leq n$, then $(l, j) \stackrel{c}{>}(l+1, j)$ and $(i, l) \stackrel{c}{>}(i, l+1)$.
3. All criticality relations on the system are the foregoing ones and those deducible from them by the transitivity property.

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