Reliability of 2-Dimensional Consecutive-k-out-of-n:F Systems

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Key Words — **2-Dimensional consecutive-k-out-of-n system, reliability bound, Weibull limit theorem.**

Reader Aids —
General purpose: Present a new result
Special math needed for explanations: Elementary probability theory
Special math needed to use results: Same
Results useful to: Reliability analysts and theoreticians.

Abstract — **This paper derives lower & upper reliability bounds for the 2-dimensional consecutive k-out-of-n:F system (Salvia Lasher, 1990) with independent but not necessarily identically distributed components. A Weibull limit theorem is proved for system time-to- failure for i.i.d. components.**

1. INTRODUCTION

The 2-dimensional (2D) consecutive-k-out-of-n:F system was introduced by Salvia & Lasher [9] by generalizing the notion of the consecutive-k-out-of-n:F system [7, 8]. It consists of a square grid of size n (containing n^2 components) and fails if and only if there is at least one square of size k $(1<k\leq n)$ whose components are all failed. The design of electronic devices, disease diagnosis, and pattern detection are some of the areas where the 2D-consecutive-k-out-of-n:F system applies.

Salvia & Lasher [9] gave bounds for the reliability of the 2D system by relating it to certain (1-dimensional) consecutive-k-out-of-n:F systems. Their bounds apply only to i.i.d. components.

This paper studies the reliability of a 2D-consecutive-k-out-of-n:F system with s-independent but not necessarily s-identical components. By applying the powerful Chen-Stein method [5] (as refined by Barbour & Eagleson [2], Barbour & Holst [3], and Arratia *et al* [1]), we approximate system reliability by an exponential term, and estimate the error of the approximation. The well known minimal-cut lower bound [4, 6] for s-coherent systems, is also applied to the system, giving an alternative approximation (from below). Our bounds which are easy to compute, provide good estimates at least for high reliability systems. We show how our results can be effectively used for proving limiting theorems for the distribution of the lifetime of large i.i.d. systems.

All proofs are in the appendix.

2. *NOTATION & ASSUMPTIONS*

Notation

- n size of square grid containing the n^2 components
- k size of square subgrid causing system failure
- p_{ij}, q_{ij} [reliability, unreliability] of component (i,j), $i,j=1,\ldots,n$
- q $\max_{1\leq i,j\leq n}\{q_{ij}\}$: failure probability of the worst component
- A_{ij} minimal cut-set with left uppermost component (i,j): $A_{ij}=\{(i+x-1,j+y-1): x=1,2,\ldots,k;\ y=1,2,\ldots,k\}$, $1\leq i,j\leq n-k+1$
- C set of all minimal cut-sets: $C=\{A_{ij}: i=1,2,\ldots,n-k+1;\ j=1,2,\ldots,n-k+1\}$
- A subset of components: $A\subseteq\{(i,j): i=1,\ldots,n; j=1,\ldots,n\}$
- q_A $\Pi_{(i,j)\in A}\, q_{ij}$: Pr{all components of A are failed}
- ϕ $$\sum_{A\in C} q_A = \sum_{i=1}^{n-k+1}\sum_{j=1}^{n-k+1} q_{A_{ij}}$$
- T_n, R_n [failure-time, reliability] of a 2D-consecutive-k-out-of-n:F system
- expf$(\cdot)$,expfc$(\cdot)$ [Cdf, Sf] of the exponential distribution
- weif$(\cdot;\beta)$ Cdf of the Weibull distribution with shape parameter β.

The p_{ij}, q_{ij}, q_A, q, ϕ, R refer to the fixed time-interval $[0,t]$. To simplify the notation, we usually write, *eg*, q_{ij}, ϕ, instead of $q_{ij}(t)$, $\phi(t)$.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

Assumptions

A. Each component and the system are either working or failed.

B. For given n, all n^2 components of the system are mutually s-independent.

C. The system fails if and only if there is at least one square grid of size k $(1<k\leq n)$ whose components are all failed.

3. STATEMENT OF RESULTS

It is very difficult (probably impossible) to derive simple explicit formulas for the reliability of a general 2D-consecutive-k-out-of-n:F system [9]. Hence, the derivation of good approximations of system reliability is valuable. Salvia & Lasher [9] proposed upper & lower bounds for the special case of i.i.d. components, based on the reliability of certain consecutive-k-out-of-n:F systems.

Our paper has 2 purposes:

- introduce some additional bounds for the reliability of a 2D-consecutive-k-out-of-n:F system, which apply to s-independent (but not necessarily s-identical) components;
- explore the asymptotic behavior of large 2D-consecutive-k-out-of-n:F systems through a Weibull limit theorem.

Theorem 1 approximates the system reliability by an exponential term, and estimates the error of the approximation.

Theorem 1. Let $n \geq k \geq 2$; let t be a fixed positive real number.

$$|\Pr\{T_n > t\} - \text{expfc}(\phi)| \leq \text{expf}(\phi) \cdot [(2k-1)^2 \cdot q^{k^2} + 4Q(k)].$$

$$Q(k) \equiv \sum_{i=1}^{k} \sum_{j=1}^{k} q^{k^2 - ij} - 1. \quad ◄$$

A slight modification of the error term of theorem 1 gives:
Corollary 1. Let $n \geq k \geq 2$; let t be a fixed positive real number.

$$|\Pr\{T_n > t\} - \text{expfc}(\phi)| \leq \text{expf}(\phi) \cdot [(2k-1)^2 \cdot q^{k^2} + 4(k^2-1) \cdot q^k]. \quad ◄$$

Theorem 1 can be used to derive lower & upper bounds of system reliability:

$$\text{LB} = \text{expfc}(\phi) - \text{expf}(\phi) \cdot [(2k-1)^2 q^{k^2} + 4Q(k)],$$

$$\text{UB} = \text{expfc}(\phi) + \text{expf}(\phi) \cdot [(2k-1)^2 q^{k^2} + 4Q(k)].$$

Numerical analysis shows that LB is, in most cases, worse than the one derived from Esary & Proschan [6: page 337, theorem 5.2]:

$$\text{LB}_1 = \prod_{i=1}^{n-k+1} \prod_{j=1}^{n-k+1} (1 - q_{A_{ij}}).$$

Hence, the most powerful approximation to the reliability of a 2D-consecutive-k-out-of-n:F system, seems to be the inequality:

$$\text{LB}_1 \leq R \leq \text{UB}.$$

Since the error term of theorem 1 becomes negligible when q approaches zero, it is apparent that LB (and therefore LB_1) and UB provide good approximations to the system reliability whenever the p_{ij} are close enough to 1, *viz*, the components are of excellent quality. Our extensive numerical calculations support this assertion. Table 1 presents LB, LB_1, UB for:

$$p_{ij} = \begin{cases} 0.7, & \text{for } i+j \text{ odd}, \\ 0.75, & \text{otherwise.} \end{cases}$$

The last column gives an upper bound of the relative error if we estimate the reliability by $(\text{LB}_1 + \text{UB})/2$. As numerical computations indicate, in the i.i.d. case and for $p_{ij} = p \geq 0.9$, the relative error bound increases with n, and decreases with p and k. Moreover, for $k > 3$, the relative error bound is less than 10^{-6} for all n. For $k = 2,3$ and $p = 0.9$, the relative error bound does not exceed 1% for $n < 47, 35000$ respectively.

TABLE 1
Comparison of Bounds

n	k	LB	LB1	UB	Error (%)
3	2	0.957707	0.977689	0.997795	1.03
5	2	0.836392	0.913699	0.991471	4.26
5	3	0.999904	0.999922	0.999941	0.00
5	4	1.000000	1.000000	1.000000	0.00
10	2	0.304369	0.633237	0.963735	26.10
10	3	0.999313	0.999443	0.999574	0.01
10	4	1.000000	1.000000	1.000000	0.00
50	3	0.975507	0.980152	0.984798	0.24
50	4	0.999998	0.999998	0.999998	0.00
50	5	1.000000	1.000000	1.000000	0.00
100	3	0.901066	0.919830	0.938595	1.02
100	4	0.999990	0.999991	0.999991	0.00
100	5	1.000000	1.000000	1.000000	0.00

The remainder of the paper examines systems with i.i.d. components:

$$q_{ij} = q, \; i = 1,2,\ldots,n; \; j = 1,2,\ldots,n.$$

All the above results still apply, and since

$$\phi = (n-k+1)^2 \cdot q^{k^2}, \; q_{A_{ij}} = q^{k^2}, \text{ for } i,j = 1,2,\ldots,n,$$

the bounds take on a slightly more convenient form. Using corollary 1 we can easily deduce the following results which refer to large $(n \to \infty)$ i.i.d. systems.

Theorem 2. Let T_n be the time to failure of a sequence of 2D-consecutive-k_n-out-of-n:F systems with i.i.d. components. If t_n, $n = 1,2\ldots$ is a sequence of positive real numbers such that:

$$\lim_{n \to \infty} (n^2 \cdot q^{k_n^2}(t_n)) = \mu, \; \lim_{n \to \infty} (k_n{}^2 \cdot q^{k_n}(t_n)) = 0.$$

Then: $\lim_{n \to \infty} (\Pr\{T_n \geq t_n\}) = \text{expfc}(\mu)$. ◄

If k is fixed $(k_n = k, \; n = 1,2,\ldots)$, then assumption #2 of theorem 2 is not necessary.

Corollary 2. Let T_n be the time to failure of a sequence of 2D-consecutive-k-out-of-n:F systems with i.i.d. components. If t_n, $n = 1,2\ldots$ is a sequence of positive real numbers such that:

$$\lim_{n \to \infty} (n^2 q^{k^2}(t_n)) = \mu.$$

Then: $\lim_{n \to \infty} (\Pr\{T_n > t_n\}) = \text{expfc}(\mu)$. ◄

Our limit theorem, under quite general assumptions on the lifetime distributions of the individual components, proves that the (properly normalized) system time to failure converges to the Weibull distribution as $n \to \infty$.

Theorem 3. Let T_n be the time to failure of a sequence of 2D-consecutive-k-out-of-n:F systems with i.i.d. components whose common failure probability function is of the form:

$q(t) = (\mu \cdot t)^a + o(t^a)$. ($a$ is a positive real number). Then

$$\lim_{n\to\infty} (\Pr\{n^{2/(ak^2)} \cdot T_n \le t\}) = \text{weif}(\mu \cdot t;\ a \cdot k^2). \qquad ◄$$

APPENDIX

A.1 Proof of Theorem 1

For every minimal cut set $A \in \boldsymbol{C}$, define a binary r.v. X_A which takes on the value 1 if and only if all components $(i,j) \in A$ fail in time less than or equal to t (and 0 otherwise), and a *neighborhood of dependence* $\boldsymbol{C}_A$ containing all minimal cut sets which have at least one common component with A:

$$C_A = \{B \in C\colon BA \ne \emptyset\}.$$

Introduce the r.v. $W = \Sigma_{A \in C} X_A$; then:

$$\Pr\{T_n > t\} = \Pr\{W=0\};\ \mathrm{E}\{W\} = \sum_{A\in C} \Pr\{X_A = 1\}$$

$$= \sum_{A\in C} q_A = \phi.$$

Since, for every $B \in C - C_A$ the r.v. X_B are s-independent of X_A, [1: theorem 1] gives:

$$|\Pr\{T_n > t\} - \text{expfc}(\phi)| \le [\text{expf}(\phi)/\phi] \cdot \sum_{A\in C} q_A$$

$$\cdot \left(\sum_{B \in C_A} q_B + \sum_{A \in C_A - \{A\}} q_{A'B} \right)$$

A' = complementary set of A.

It is not difficult to verify that:

$$\sum_{B\in C_A} q_B \le (2k-1)^2 q^{k^2},$$

$$\sum_{B \in C_A - \{A\}} q_{A'B} \le 4\left\{ \sum_{i=1}^{k} \sum_{j=1}^{k} q^{k^2 - i\cdot j} - 1 \right\}. \qquad Q.E.D.$$

A.2 Proof of Corollary 1

For $(i,j) \ne (k,k)$ we have $k^2 - i \cdot j \ge k$; thus:

$$q^{k^2 - i \cdot j} \le q^k. \qquad Q.E.D.$$

Proof of Theorem 2

$$\lim_{n\to\infty} (\phi(t_n)) = \lim_{n\to\infty} (n - k_n + 1)^2 \cdot q^{k_n^2}(t_n) = \mu,$$

$$\lim_{n\to\infty} ((2k_n - 1)^2 \cdot q^{k_n^2}(t_n)) = \lim_{n\to\infty} ((k_n{}^2 - 1) \cdot q^{k_n}(t_n)) = 0.$$

Apply corollary 1 in the time interval $[0,t_n]$; account for the last relation. Then the error term of the approximation tends to 0 as $n \to \infty$, and therefore:

$$\lim_{n\to\infty} (\Pr\{T_n > t_n\}) = \lim_{n\to\infty} (\text{expfc}(\phi(t_n)) = \text{expfc}(\mu). \quad Q.E.D.$$

A.3 Proof of Corollary 2

It follows immediately from corollary 1 or directly from theorem 2. *Q.E.D.*

Proof of Theorem 3

If $t_n = t \cdot n^{-2/(ak^2)}$ it is evident that:

$$q(t_n) = (\mu \cdot t)^a \cdot n^{-2/k^2} + n^{-2/k^2} \cdot o(1)$$

Therefore:

$$n^2 q^{k^2}(t_n) = (\mu \cdot t)^{a \cdot k^2} + o(1),$$

and the proof is easily completed using corollary 2. *Q.E.D.*

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