Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 30, No. 6, pp. 3345–3348, 1997
Proc. 2nd World Congress of Nonlinear Analysts

0362-546X/97 $17.00 + 0.00

PII: S0362-546X(97)00254-X

A RELIABILITY BOUND FOR 2-DIMENSIONAL CONSECUTIVE k-OUT-OF-n: F SYSTEMS

M. V. KOUTRAS,† G. K. PAPADOPOULOS‡ and S. G. PAPASTAVRIDIS†

†Department of Mathematics, University of Athens, Panepistemiopolis, 15710 Athens, GREECE

‡Greek Pedagogical Institute, 396 Mesogeion St., Agia Paraskevi, 15341 Athens, GREECE

Key words and phrases: 2-Dimensional consecutive k-out-of-n:F system, reliability bound, Poisson approximation, Compound Poisson approximation, Chen-Stein method, minimal cut set.

1. INTRODUCTION

The 2-dimensional consecutive-k-out-of-n:F system was introduced by Salvia & Lasher [1] by generalizing the notion of the consecutive-k-out-of-n:F system [2,3]. It consists of a square grid of size n (containing n^2 components) and fails if and only if there is at least one square grid of size $k\,(1 < k \le n)$ whose components are failed. This system has recently received extensive research interest, a fact mainly due to its applicability in various areas such as safety monitoring systems, design of electronic devices, disease diagnosis and pattern detection.

For a coherent system whose minimal cut sets have been specified, the classical lower bound is the one obtained by Esary & Proschan [4] (see also Barlow & Proshan [5]). In a 2-dimensional consecutive-k-out-of-n:F system the minimal cut sets consist of k^2 components placed on a rectangular kxk grid. Therefore, denoting by p_{ij} $(q_{ij} = 1 - p_{ij})$ the survival (failure) probability of system's components we deduce the following lower bound

$$LB_{E\&P} = \prod_{i=1}^{n-k+1} \prod_{j=1}^{n-k+1} \Big(1 - \prod_{\mu=i}^{i+k-1} \prod_{\nu=j}^{j+k-1} q_{\mu\nu}\Big)$$

Salvia & Lasher [1] suggested an alternative lower bound (obtained by employing a "binomial type" argument) whereas Koutras, Papadopoulos & Papastavridis [6] used the celebrated Chen-Stein method to approximate a 2-dimensional consecutive-k-out-of-n:F system's reliability by an exponential type lower bound. Finally, Barbour, Chryssaphinou & Roos [7] gave a compound Poisson local Chen-Stein lower bound for the same quantity. Extensive numerical experimentation revealed that the classical Esary & Proschan [4] lower bound is in general performing quite well and in view of its simplicity is certainly preferable over all other bounds.

The first upper bound for a 2-dimensional consecutive-k-out-of-n:F system's reliability was the one given by Salvia & Lasher [1]. As pointed out by Ksir [8] later, this bound was incorrect and although the error for high reliability systems is negligible, it can only serve as an approximation and not as an upper bound. Several upper bounds were subsequently proposed by Koutras, Papadopoulos & Papastavridis [6], Fu & Koutras [9], [10], Barbour, Chryssaphinou & Roos [7].

The purpose of the present work is to present a simple upper bound which is expressed in terms of a 1-dimensional consecutive-k-out-of-n :F system. Its derivation is based on the chain rule of probabilities and makes effective use of the specific structure of the system.

2. STATEMENT OF RESULTS

Before advancing to the statement and proof of our main result let us introduce the necessary notations and give two lemmas which are instrumental for the development of our upper bound.

We denote by Δ_{in} the event that in the system consisting of rows $i-k+1,\ldots,i\,(i \geq k)$ and columns $1,2,\ldots,n$, there is no grid of size kxk (or greater rectangular) with all its components down. Let also G_i be the event that in row $i-k\,(i \geq k+1)$ there are no k or more consecutive components, which are failed and B_i the event that in the system consisting of rows $1,2,\ldots,i\,(i \geq k+1)$ and columns $1,2,\ldots,n$, there is no grid of size kxk or greater with all its components down. In the sequel, we shall use the notation A' for the complement of any event A.

LEMMA 2.1. If A, B, C are three arbitrary events, then

$$\Pr\{A|B\} \geq \Pr\{A|BC\} \cdot \Pr\{C|B\}.$$

PROOF. Manifestly

$$\Pr\{A|BC\}\Pr\{C|B\} = \frac{\Pr\{ABC\} \cdot \Pr\{CB\}}{\Pr\{BC\} \cdot \Pr\{B\}} = \frac{\Pr\{ABC\}}{\Pr\{B\}} \leq \frac{\Pr\{AB\}}{\Pr\{B\}} = \Pr\{A|B\}.$$

LEMMA 2.2. For all $i \geq k+1$, we have

a. $\Pr\{G_i|B_{i-1}\} \geq \Pr\{G_i\}$

b. $\Pr\{B_i'|B_{i-1}G_i\} = \Pr\{\Delta_{in}'\}$.

PROOF. Part (a) is obvious. To prove part (b) it suffices to observe that

$$\Pr\{B_i'|B_{i-1}G_i\} = \frac{\Pr\{B_i' B_{i-1} G_i\}}{\Pr\{B_{i-1}G_i\}} = \frac{\Pr\{\Delta_{in}' B_{i-k} G_i\}}{\Pr\{B_{i-k}G_i\}}$$

and make use of the independency of events Δ_{in}', $B_{i-k}G_i$.

THEOREM 2.1. Let $2 \leq k \leq n$ and t be a fixed positive real number. For the system's life time T we have,

$$\Pr\{T > t\} \leq \Pr\{\Delta_{kn}\} \prod_{i=k+1}^{n} (1 - \Pr\{\Delta_{in}'\}\Pr\{G_i\}).$$

PROOF. It is evident that the sequence B_i, $i = k, k+1, \ldots, n$ is monotone decreasing and therefore

$$\Pr\{T > t\} = \Pr\{B_n\} = \Pr\{B_k\} \prod_{i=k+1}^{n} \Pr\{B_i|B_{i-1}\} = \Pr\{B_k\} \prod_{i=k+1}^{n} (1 - \Pr\{B_i'|B_{i-1}\}). \tag{2.1}$$

Moreover, Lemma 2.1 yields

$$\Pr\{B_i'|B_{i-1}\} \geq \Pr\{B_i'|B_{i-1}G_i\}\Pr\{G_i|B_{i-1}\}. \tag{2.2}$$

Theorem 2.1 follows now immediately by making use of Lemma 2.2 and (2.1)-(2.2).

It is noteworthy that the probability $\Pr\{\Delta_{in}\}$, $i=k,k+1,\ldots n$, is the reliability $R(k,n,Q_{ij})$ of a linear 1-dimensional consecutive-k-out-of-n:F system with component's failure probabilities

$$Q_{ij} = \prod_{\mu=i-k+1}^{i} q_{\mu j} \quad j=1,2,\ldots,n$$

and $\Pr\{G_i\}$, is the reliability $R(k,n,q_{i-k,j})$ of a linear 1-dimensional consecutive-k-out-of-n:F system with component's failure probabilities $q_{i-k,j}$, $j=1,2,\ldots,n$.

Therefore, Theorem 2.1 could be restated as

COROLLARY 2.1. Let R be the reliability of a 2-dimensional consecutive-k-out-of-n:F system with independent but not necessarily identically distributed components. Then

$$R \le R(k,n,Q_{kj}) \cdot \prod_{i=k+1}^{n} \{1-[1-R(k,n,Q_{ij})]\cdot R(k,n,q_{i-k,j})\} = UB .$$

Our extensive numerical experimentation indicated that in most cases our upper bound provides quite a good approximation of the actual system's reliability. No direct theoretical comparison is available for the upper bounds proposed so far, i.e. our bound (UB), Koutras et al's [6] bound (UB_{C-S}), Fu & Koutras's [9] bound ($UB_{F\&K}$), and Barbour et al's [7] bound (UCP).

Tables 1 and 2 present values of $LB_{E\&P}$, UB, UB_{C-S}, $UB_{F\&K}$, UCP for several of n and k, and component reliabilities

$$I.\ p_{ij} = \begin{cases} 0.7 & if\ \ i+j\ \ odd \\ 0.75 & if\ \ i+j\ \ even \end{cases} \qquad II.\ p_{ij} = \begin{cases} 0.5 & if\ \ |i-j| \le 1 \\ 1-\dfrac{1}{|i-j|} & if\ \ |i-j| > 1 \end{cases}$$

respectively. We mention that in these cases where the Chen-Stein upper bounds exceeded 1, the respective entry in the table was set 1.

Table 1. Lower and upper bounds for the reliability of a 2-dimensional consecutive-k-out-of-n:F system with

$$p_{ij} = \begin{cases} 0.7 & if\ \ i+j\ \ odd \\ 0.75 & if\ \ i+j\ \ even \end{cases}$$

n	k	$LB_{E\&P}$	UB	$UB_{F\&K}$	UB_{C-S}	UCP
3	2	0.9777	0.9799	0.9834	0.9978	0.9804
5	2	0.9137	0.9319	0.9448	0.9915	0.9362
5	3	0.9999	0.9999	0.9999	0.9999	0.9999
10	2	0.6332	0.7723	0.7708	0.9637	0.7773
10	3	0.9994	0.9995	0.9997	0.9996	0.9995
10	4	1.0000	1.0000	1.0000	1.0000	1.0000
20	2	0.1305	0.5410	0.3292	0.9139	0.6907
20	3	0.9972	0.9979	0.9984	0.9978	0.9973
20	4	1.0000	1.0000	1.0000	1.0000	1.0000
50	2	0.0000	0.4788	0.0007	0.9009	1.0000
50	3	0.9802	0.9907	0.9893	0.9848	0.9811
50	4	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2. Lower and upper bounds for the reliability of a 2-dimensional consecutive-k-out-of-n:F system with

$$p_{ij} = \begin{cases} 0.5 & if \quad |i-j| \le 1 \\ 1 - \dfrac{1}{|i-j|} & if \quad |i-j| > 1 \end{cases}$$

n	k	$LB_{E\&P}$	UB	$UB_{F\&K}$	UB_{C-S}	UCP
3	2	0.7725	0.8297	0.8661	1.0000	0.9185
5	2	0.4302	0.6635	0.7156	1.0000	1.0000
5	3	0.9881	0.9904	0.9942	1.0000	0.9904
10	2	0.0872	0.4851	0.4342	1.0000	1.0000
10	3	0.9611	0.9743	0.9851	1.0000	0.9717
10	4	0.9999	0.9999	1.0000	1.0000	0.9999
20	2	0.0035	0.2948	0.1567	1.0000	1.0000
20	3	0.9091	0.9447	0.9672	1.0000	0.9360
20	4	0.9998	0.9998	0.9999	0.9999	0.9980
50	2	0.0000	0.0742	0.0074	1.0000	1.0000
50	3	0.7695	0.8617	0.9154	1.0000	0.8404
50	4	0.9993	0.9994	0.9997	0.9997	0.7160

REFERENCES

1. SALVIA, A. A. & LASHER, W. C., 2-Dimensional consecutive-k-out-of-n:F models, *IEEE Trans. Reliability* **39**, 382-385 (1990).
2. CHAO, M. T., FU, J. C. & KOUTRAS, M. V., Survey of the reliability studies of consecutive-k-out-of-n:F and related systems, *IEEE Trans. Reliability* **44**, 120-127 (1995).
3. PAPASTAVRIDIS, S. G. & KOUTRAS, M. V., Consecutive-k-out-of-n systems, in *New trends in system reliability evaluation*, (K. B. Misra Ed), Elsevier/North Holland, Amsterdam, 228-248 (1993).
4. ESARY, J. D. & PROSHCAN, F., A reliability bound for systems of maintained interdependent components, *J. Amer. Statistical Assoc.* **65**, 329-339 (1970).
5. BARLOW, R. E. & PROSHCAN, F., Statistical theory of reliability and life testing, *To Begin With, Silver Spring* (1981).
6. KOUTRAS, M. V., PAPADOPOULOS, G. K. & PAPASTAVRIDIS, S. G., Reliability of 2-Dimensional Consecutive-k-out-of-n:F systems, *IEEE Trans. Reliability* **42**, 658-661 (1993).
7. BARBOUR, A. D., CHRYSSAPHINOU, O. & ROOS, M., Compound Poisson approximation in systems reliability, *Naval Research Logistics* **43**, 251-264 (1996).
8. KSIR, B., Comment on: 2-Dimensional consecutive-k-out-of-n:F models, *IEEE Trans. Reliability* **41**, 575 (1992).
9. FU, J. C. & KOUTRAS, M. V., Poisson approximations for 2-dimensional patterns, *Ann.Inst.Statist. Math.* **46**, 179-192 (1994).
10. FU, J. C. & KOUTRAS, M. V., Reliability bounds for coherent structures with independent components, *Statist. & Prob. Letters* **22**, 137-148 (1995).